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a dubious and a less dubious case

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Artificial Language in Ancient Mesopotamia – a Dubious and a Less Dubious Case

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Babylonian mathematical symbols?

In the preface to his *Mathematische Keilschrift-Texte*, Neugebauer [MKT I: viii] explained his choice not to expand the logograms used in the mathematical texts into grammatical Akkadian in the following way:

> Erstens scheint es mir ein *methodischer* Fehler zu sein, die Aufgabe der Textreproduktion mit der in vielen Fällen noch gänzlich ungeklärten Frage nach den akkadischen Äkvivalenten der ideographischen Schreibungen zu belasten [...]. Zweitens zerstört man durch die Elimination der Ideogramme eine der geschichtlich wichtigsten Züge unserer Texte, nämlich die Existenz einer mathematischen *Symbolik*. Diese Anschauung ist in glänzender Weise durch die Ausdrucksweise der Serientexte [...] bestätigt worden. Solche Texte akkadisieren heißt ungefähr soviel, wie $\sin^2\alpha+\cos^2\alpha = 1$ durch "viereckiger Busen von α vermehrt um den viereckigen Mitbusen von α ist gleich eins" zu umschreiben.

The first part of the argument has been amply confirmed since then, but it has hardly been taken note of (and had little chance of being understood) outside the community of Assyriologists. The second part – the identification of the logograms with a mathematical symbolism – has gained some currency in the general mathematico-historical literature.

The passage in MKT is too succinct to give much insight in what precisely was on Neugebauer's mind – and the polemical turn in the end does not help much. His *Vorgriechische Mathematik* [1934: 68–72] is much more explicit:

> [...] Jedes algebraische Operieren setzt voraus, daß man sowohl für die mathematischen Operationen wie für die Größen gewisse feststehende Symbole besitzt. Erst die Existenz einer solchen Begriffsschrift macht es möglich, daß man Größen, die nicht numerisch benannt sind, miteinander kombiniert und neue Kombinationen aus ihnen herleitet.
>
> *Eine solche Symbolschrift ist aber von selbst beim Schreiben akkadischer Texte gegeben.* Wie wir gesehen haben, hat man nämlich dabei zweierlei Ausdrucksmittel zur Verfügung: entweder bedient man sich der silbenschriftlichen Schreibweise, oder aber man schreibt mit Ideogrammen. Beide Schreibweisen wechseln in den meisten akkadischen Texten ununterbrochen und ganz willkürlich miteinander ab. Dieses rein geschichtlich entstandene Verfahren ist nun für die mathematische Terminologie von grundsätzlicher Bedeutung. Es hat sich nämlich dort ganz entsprechend das Verfahren eingebürgert, die mathematischen Begriffe, d.h. sowohl Operationen wie Größen, ideographisch zu schreiben. Das bedeutet also, daß in einem akkadisch geschriebenen Text gerade die entscheidenden Begriffe immer mit Hilfe konventioneller Einzelsymbole geschrieben werden können. So hat man also von Anfang an über die wichtigste Grundlage für eine algebraische Entwicklung, nämlich eine geeignete Symbolik, verfügt. Zunächst bedeutet dies sicherlich nichts anderes als die auch sonst in der Schrift übliche Art, willkürlich zwischen Ideogrammen und syllabischen

Schreibungen zu wechseln. Gerade für das Mathematische muß aber die Existenz von konventionellen Einzelsymbolen nach Art der Ideogramme ganz von selbst von größter praktischer Bedeutung für die leichte Übersehbarkeit der Operationen werden. So muß alles zu einer reinen Formelschrift treiben, und wir werden sogleich unten an Beispielen sehen, daß dies in der Tat eingetreten ist.[1]

Diese Tatsache ist nun beim Lesen mathematischer Keilschrifttexte von größter Bedeutung. Kennt man nur eine gewisse Anzahl immer wiederkehrender Ideogramme für die einzelnen Operationen, für Termini wie Summe, Differenz usw., für Länge, Breite, Durchmesser, so läßt sich ein solcher Text direkt in unsere Formelsprache umschreiben, ohne daß man dabei zu wissen braucht, wie im Akkadischen diese Ideogramme ausgesprochen worden sind. So wird beispielsweise ein gewisses Ideogramm RI immer für den Begriff einer Linie in einer Figur verwendet, die zwei Teilbereiche voneinander trennt (z. B. die Sehne eines Kreises); ohne daß die akkadische Lesung dieses Ideogramms bekannt war, konnte man immer die mathematische Funktion dieser Größe in den Rechnungen richtig verstehen. Erst viel später haben syllabisch geschriebene Parallelstellen gezeigt, daß RI als *pi-ir-kum* gelesen wurde, was soviel wie “Riegel” bedeutet. Damit ist dann gleichzeitig ein Einblick in die Bedeutungsgeschichte dieses Terminus gewonnen, aber für das Verständnis seiner Rolle in den mathematischen Texten ist die Kenntnis der Aussprache ebenso gleichgültig, wie man heute nicht zu wissen braucht, wie die mathematischen Symbole z. B. in einer russisch geschriebenen Arbeit ausgesprochen werden.

Wir wollen diese Erscheinungen nun an zwei Einzelbeispielen aus mathematischen Texten besprechen. Das erste entstammt einem alten, relativ wenig ideographisch geschriebenen Text. Das zweite ist rein ideographisch geschrieben und wird zeigen, daß es sich hier nur noch um eine reine Formelsprache handelt[2]. [...]

[Dieses] zweite Beispiel stammt aus [dem Text YBC 4709]. Als Beispiel für seine Terminologie sei etwa die Aufgabengruppe Nr. 4 bis 7 dieses Textes herausgegriffen. Dieser Textabschnitt lautet:

Nr. 4 uš a-rá 3 e-tab
sag a-rá 2 e-tab
gar-gar íb-si$_8$
a-šá us daḫ-ma 4,56,40

---

[1] Einige Texte zeigen übrigens, daß man noch auf eine andere Weise bewußt zu solchen konventionellen Vereinfachungen hinstrebte, nämlich durch Verwendung wirklicher Abkürzungen für akkadische Worte, dadurch, daß man nur eine Silbe schrieb. Schriftgeschichtlich gesehen werden hier also als Silben verwendete Ideogramme an Stelle von Ideogrammen verwendet!

[2] Übrigens werden in den mathematischen Texten sumerische Bildungen oft in grammatisch vollständig unkorrekter Weise verwendet, was zeigt, daß sie wirklich nur noch reine Symbole für Begriffe waren, deren ursprüngliche Ableitung aus einer anderen gesprochenen Sprache vollständig ignoriert worden ist.

Nr. 5 a-šá uš a-rá 2 e-tab
daḫ-ma 5,11,40

Nr. 6 a-šá uš ba-zi 4,26,40

Nr. 7 a-rá 2 e-tab
ba-zi-ma 4,11,40

Es ist schwer, ihn “wörtlich” zu übersetzen, denn er enthält eigentlich keinerlei grammatische Struktur mehr, sondern fast nur noch Ideogramme, deren Wiedergabe durch ein Wort in unserer Sprache insofern inkorrekt ist, als wir notwendigerweise eine bestimmte grammatische Form wählen müssen, während sie in Wirklichkeit nicht mehr in einem solchen Text liegt. Man kann unseren Text ungefähr folgendermaßen reproduzieren:

Nr. 4 Länge mit 3 vervielfacht
Breite mit 2 vervielfacht
addiert quadratisch
Fläche (der) Länge addiert und so 4,56,40

Nr. 5 Fläche (der) Länge mit 2 vervielfacht
addiert und so 5,11,40

Nr. 6 Fläche (der) Länge subtrahiert 4,26,40

Nr. 7 mit 2 vervielfacht
subtrahiert und so 4,11,40

Man kann diese Aufgaben sehr viel sachgemäßer direkt in eine Formelsprache übersetzen. Wir brauchen nur für die Ideogramme der Unbekannten uš “Länge” bzw. sag “Breite” die Zeichen x bzw. y einzusetzen und Entsprechendes für die Operationen. Man erhält dann, wenn man die Reihenfolge beibehält, die folgende “Übersetzung”, die dem wirklichen Textzustand am nächsten kommt:

Nr. 4 $x \cdot 3$
$y \cdot 2$
$+^2$
$x^2 + = 4{,}56{,}40$

Nr. 5 $x^2 \cdot 2$
$+ = 5{,}11{,}40$

Nr. 6 $x^2 - 4{,}26{,}40$

Nr. 7 $\cdot 2$
$- = 4{,}11{,}40$

> In diesen Formeln sind immer zunächst die zu kombinierenden Größen genannt und dann die Operationen, die mit ihnen auszuführen sind [...]. Sieht man von dieser Äußerlichkeit ab, so entsprechen unsere Beispiele genau den folgenden Formeln:
>
> | | |
> |---|---|
> | Nr. 4 | $(3x+2y)^2+x^2 = 4,56,40$ |
> | Nr. 5 | $+2x^2 = 5,11,40$ |
> | Nr. 6 | $-x^2 = 4,26,40$ |
> | Nr. 7 | $-2x^2 = 4,11,40.$ |
>
> Wir sehen also, daß es sich hier um eine im Grunde vollständig algebraische Ausdrucksweise handelt, in der also vor allem Kombinationen der unbekannten Größen gebildet werden.

Most of what Neugebauer says in these pages about the character of mathematical cuneiform is still valid after almost 70 years (and in 1934 Assyriology was not much older than 70 years!).[3] We might add that the shift between logographic and syllabic writing in ordinary text turns out to be somewhat less arbitrary than Neugebauer believes when closed text groups are examined; but this is a minor problem. The claim that the use of logograms in mathematical texts increases with time is also problematic, as is the assertion that ideograms are used preferentially for the mathematical terms in texts that also contain non-mathematical language. The most important revision that modern Assyriologists would apply in this respect is probably that they would speak of *logograms*, word signs, and not of *ideograms*, signs for translinguistic operations and concepts. It should also be remarked that modern mathematical symbols are clearly distinguishable from ordinary words: some of them, for instance +, Σ, Π and ∫ only exist as mathematical symbols; others are strings of letters that do not correspond to the writing of words from spoken language (exp, cos, etc.). In contrast, most babylonian word signs, including those designating mathematical operations and "variables", also serve as phonetic syllabic signs, or possess a plurality of logographic interpretations; within texts belonging to a particular text group, such ambiguities are less outspoken, but they do not disappear. The use of mathematical logograms thus did not produce quite the same *leichte Übersehbarkeit der Operationen* as do modern mathematical symbols – but since Neugebauer was deeply engaged in the interpretation of the whole corpus by then, we may

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[3] A third discussion of the same topic is found in [Neugebauer 1932: 5]:

> [es] darf nicht übersehen werden, daß diese Ideogramme praktisch den Wert mathematischer *Formelzeichen* besitzen und so eine relativ sehr übersichtliche Ausdrucksweise des Rechnungsgangs gestatten – genau so, wie unsere heutigen Symbole ursprünglich aus Wortzeichen entstanden sind, die man heute in jeder Sprache irgendwie ausspricht.

trust him as a witness that they did facilitate task.

On the whole, Neugebauer's claims as set forth here were thus convincing, and to a large extent they stay so. But we should take care not to read more into the claims than was intended by their author. Neugebauer referred to two characteristics of symbolic writing, and to these two only:

– its existence "macht es möglich, daß man Größen, die nicht numerisch benannt sind, miteinander kombiniert und neue Kombinationen aus ihnen herleitet";
– "die Existenz von konventionellen Einzelsymbolen nach Art der Ideogramme" ensures "die leichte Übersehbarkeit der Operationen".

Nesselmann's scheme

In the historiography of mathematics, in particular when the development of algebra is discussed, a different aspect of the notion of symbolism is emphasized. The conventional distinction between "rhetorical", "syncopated" and "symbolic" algebra goes back to Nesselmann [1842: 302]. In "rhetorical algebra", his "erste und niedrigste Stufe", everything in the calculation is explained in full words. "Syncopated algebra" makes use of standard abbreviations for certain recurrent concepts and operations, even though its exposition remains essentially rhetorical. "Symbolic algebra" is the type

> welche alle vorkommenden Formen und Operationen durch eine vollkommen ausgebildete, vom mündlichen Vortrage ganz unabhängige Zeichensprache darstellt, wodurch sie jede rhetorische Darstellung unnütz macht. Wir können eine algebraische Entwickelung von Anfang bis zu Ende völlig verständlich durchführen, ohne irgend ein geschriebenes Wort zu gebrauchen, und setzen wirklich, wenigstens bei einfacheren Entwickelungen, nur hin und wieder eine Conjunction zwischen die Formeln, um dem Leser das Suchen und Zurücklesen dadurch zu Ersparen, daß wir gleich auf die Verbinding der Formel mit dem vorhergehenden und nächstfolgenden hinweisen.

This is clearly the type to which post-Descartes algebra belongs. But Nesselmann goes on with these considerations:

> Indeß sind wir Europäer seit der Mitte des siebzehnten Jahrhunderts nicht die Ersten gewesen, sondern die *Indischen* Mathematiker sind uns hier um viele Jahrhunderte vorausgeeilt.

According to the former of these two quotations, symbolic algebra thus allows all operations to be made directly at the level of the symbolic notation; its notation constitutes an alternative to language – in a loose sense, an artificial language.

My reason to claim this sense to be only loose or approximate is that "natural" or genuine language (apart from its phatic, jussive and sundry other functions) serves to describe or refer to something external to itself. In a sequence of symbolic algebraic operations we may maintain that what goes on is not external to the symbolic operations but *nothing but* these operations.[4] The symbolic manipulations correspond to the succession of geometric operations in a Euclidean construction, not to the words describing these more or less completely.

Leaving aside these considerations for a moment we may look at some examples. A piece of indubitable rhetorical algebra can be found in al-Khwārizmī's *Algebra* – I quote Gherardo da Cremona's translation [ed. Hughes 1986: 250]:[5]

> Quod si aliquis interrogans quesierit et dixerit: "Divisi decem in duas partes. Deinde multiplicavi unam earum in alteram et provenerunt viginti unum". Tu ergo iam scivisti quod una duarum sectionum decem est res. Ipsam igitur in decem, re excepta, multiplica, et dicas: "Decem excepta re in rem sunt decem res, censu diminuto, que equantur viginti uno". Restaura igitur decem excepta re per censum, et adde censum viginti uno; et dic: "Decem res equantur viginti uno et censui". Radices ergo mediabis et erunt quinque. Quas in se multiplicabis et proveniet viginti quinque. Ex eo itaque prohice viginti unum, et remanet quattuor. Cuius accipe radicem que est duo, et minue eam ex medietate rerum. Remanet ergo tres qui est una duarum partium.

As we see, this is really a description in words of a sequence of numerical operations, referring first to a specific problem that gives rise to the operations ("Divisi decem in duas partes. Deinde multiplicavi unam earum in alteram et

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[4] I do not assert that this is the case, since this depends on the philosophy of mathematics which we apply: according to a constructivist point of view it is likely to be true, from a "Platonist" stance it is certainly a mistake; but the mere possibility to identify the symbolic operations with the thing itself distinguishes the function of mathematical symbols from that of language proper.

The conjunctions to which Nesselmann refers thus play a role beyond the one he ascribes to them: they bridge the gap between the symbolic level and its description. When the sequence is read aloud in a mathematical classroom, conjunctions and formulae together, the linguistic interpretation of the formulae will often be approximate, but having the formula before the eyes allows the student to understand the precise reference.

This piece of philosophical hairsplitting may seem inane at present; however, it will serve below.

[5] Reasons are given in [Høyrup 1998] that this translation is a better witness of al-Khwārizmī's original than the published Arabic text from which all modern translations are derived.

provenerunt viginti unum") and later to a standard problem which is solved by means of a standard algorithm ("Decem res equantur viginti uno et censui"); just as the solving operations, these are explained in full language.

Nesselmann's example of syncopated algebra is Diophantos's *Arithmetica*. A passage from I.9 [ed. Tannery 1893: I, 26] runs as follows:

> *Τετάχθω ὁ ἀφαιρούμενος ἀφ' ἑκατέρου ἀριθμοῦ, ς ᾱ. κἂν μὲν ἀπὸ τοῦ ρ̄ ἀφαιρεθῇ, λοιπαὶ Μ̊ ρ̄ ⋔ ς ᾱ· ἐὰν δὲ ἀπὸ τοῦ κ̄, λοιπαὶ Μ̊ κ̄ ⋔ ς ᾱ. καὶ δεήσει τὰ μείζονα τῶν ἐλασσόνων εἶναι ϛ$^{\pi\lambda.}$· ϛ$^{\kappa\iota\varsigma}$ ἄρα τὰ ἐλάσσονα ἴσα ἐστὶ τοῖς μείζοσιν, ϛ$^{\kappa\iota\varsigma}$ δὲ τὰ ἐλάσσονα ποιεῖ Μ̊ ρκ̄ ⋔ ς ϛ̄· ταῦτα ἴσα Μ̊ ρ̄ ⋔ ς ᾱ.*

Tannery translates this as follows:

> Ponatur subtrahendus ab utroque numero = x; si a 100 aufertur, remanent 100–*x*, si a 20, remanent 20–*x*, et oportebit maiorem residuum minoris esse 6$^{\text{plum}}$. 6$^{\text{ies}}$ igitur minor aequalis erit maiori; sed 6$^{\text{ies}}$ minor facit 120–6*x*, quae aequentur 100–*x*.

In Ver Eecke's French translation [1926: 15], the same passage runs:

> Que le nombre à retrancher de chaque nombre soit 1 arithme. Si on le retranche de 100, il reste 100 unités moins 1 arithme; et si on le retranche de 20, il reste 20 unités moins 1 arithme. Or, il faut que le plus grand reste soit le sextuple du plus petit; donc, six fois le plus petit seront égaux au plus grand. Or, six fois le plus petit donnent 120 unités moins 6 arithmes, ce qui sera égal à 100 unités moins 1 arithme.

The latter translation corresponds to that language expression into which the Greek text would be expanded when read aloud; since no operations are performed on the seemingly modern symbols in Tannery's version, this version corresponds better to the way the Greek text functioned for its readers – exactly because it makes use of symbols to which we are accustomed, it facilitates Neugebauer's "leichte Übersehbarkeit der Operationen".[6]

Some of the late medieval *abbaco* algebras are purely rhetorical – thus the

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[6] Perhaps it does so more thoroughly than warranted; as observed by Reviel Netz [2002], Tannery uses the abbreviations more systematically than any of the manuscripts he has inspected, and none of these agree with the others. But on the whole the Latin translation is faithful to Diophantos's spirit as long as we do not go beyond it and begin performing operations on the symbolic expressions.

very first specimen, written by Jacopo da Firenze in 1307 [ed. Høyrup 2000].[7] Soon, however, the beginning of symbolism proper creeps in. One example – which, if the late fourteenth-century manuscript corresponds to the original supposed to be from 1344, is the earliest European example we know – is from Dardi's *Aliabraa argibra*. I quote the Venetian text in [Vatican MS Chigiana VIII 179, fol. 5^r–5^v]

> Se tu vuo moltiplicar numero m̃ ℞, mettemo che tu vuo moltiplicar 3 m̃ ℞ de 5 via 4 m̃ ℞ de 7, tu die inprima moltiplicar li numeri l'un per l'altro, zoè 3 via 4, monta 12. E salva da parte. E po moltiplica li numeri in croce per le ℞ ch'è m̃, e quello che ne vegniva sera ℞ m̃, adunqua moltiplica 3 via m̃ ℞ de 7, monta ℞ de 63 et è m̃. Po moltiplica 4 via m̃ ℞ de 5, monta ℞ de 80 m̃, po zonzi insenbre queste 2 ℞ ch'è m̃, et averà ℞ de 63 m̃ e ℞ de 80 m̃, le qual ℞ ch'è m̃ se vorrà trazer dela $ dele moltiplication che fa più. Ora sapi che tu die moltiplicar le 2 ℞ ch'è m̃ l'una coll'altra, e farà più ℞, la qual moltiplication zonzi al numero che fa la moltiplication dei numeri l'un per l'altro. Adunqua moltiplica ℞ de 5 ch'è m̃ via ℞ de 7 ch'è m̃, e fa ℞ de 35 ch'è più, la qual ℞ zonzi ala moltiplication dei numeri, zoè a 12, et averà 12 e ℞ de 35. Ora trazi le do ℞ che fo decto davanti ch'è m̃, zoè ℞ de 63 e ℞ de 80, dela *S* dele moltiplication che fa più, e averà 12 e ℞ de 35 m̃ ℞ de 63 e m̃ ℞ de 80. E tanto monta a moltiplicar 3 m̃ ℞ de 5 via 4 m̃ ℞ de 7. E sappi che tanto valerave a cominzar dale ℞ e si m̃ dai numeri como incomenzar dai numeri e finir ale ℞, recordandote senpre de zonzere le moltiplication che fa insenbre innanzi che tu incomenzi a trazer numero over ℞ alguna de algune moltiplication per aver dela decta moltiplication più perfecto intendimento.

3 m̃ ℞ de 5
12 e ℞ de 35 m̃ ℞ de 63 e m̃ ℞ de 80
3 m̃ ℞ de 5

In this text, as we see, a first only slightly syncopated verbal explanation (℞ replaces "radice", m̃ is used for "meno", $ for "somma") is followed by a scheme which cannot be expanded into spoken phrases but only explained (as done in the preceding lines, with their reference to cross-multiplication etc.), and in which operations to be performed are hinted at by means of lines. Within the spoken text we also find a slight tendency to deviate from ordinary language and use

[7] The only manuscript in which this algebra section of Jacopo's *Tractatus algorismi* survives (Vat. Lat. 4826) goes so far in avoiding syncopation that it even abstains from using the usual medieval abbreviations in the mathematical terminology – in the algebra, *meno* is always written in full; in the coin list, it appears as m̃. The explanation of this peculiarity may be that the author was aware of introducing something new, where full transparency was therefore important (the extant manuscripts is a library copy produced by a scribe who seems to have taken great care to conserve even original spellings).

phrases which correspond to the technical operations of algebra (“le 2 ℞ ch’è m̃” instead of “le 2 ℞ che vuo trazer” or something similar).[8]

This beginning of formal, non-linguistic operations is no mere extension of the syncopation principle. This can be seen in another *abbaco* treatise, dating from c. 1365 (*Trattato dell’alcibra amuchabile*), which introduces formal operations without syncopation. An interesting passage [ed. Simi 1994: 41*f*; emphasis added] runs as follows

> Uno partì 100 in una quantità e poi partì 100 in più 5 che prima e, giunti questi due avenimenti insieme, fecie 20. Vo’ sapere in che 100 si partì in prima ed in che si partì poscia. Poni che tu partissi 100 in una cosa, vienne 100 partito per una cosa. E poi dicie che parti 100 in più 5 che prima, dunque ti conviene partire 100 in una cosa e 5, vienne 100 partito per una cosa e 5. [...] Adunque torniamo alla nostra ragione. Chaviamo 100 partito per una cosa e 100 partito per una cosa e più 5 e però poni questi due partimenti sì *come fosse uno rotto*, a questo modo come vedi disegniate qui apresso. Ed ora *multipricha in crocie*, così come faciesti dinanzi, cioè 100 vie una cosa, che fa 100 cose. Ora multipricha l’altra schisa, cioè 100 vie una cosa e 5, fanno 100 cose e 500 numeri; giungni con 100 cose, ài 200 cose e più 500 numeri. Ora multipricha ciò ch’ai di sotto alle verghe l’uno contro a l’altro, cioè una cosa via una cosa e più 5, fanno uno cienso e più 5 cose. Ora multipricha gli avenimenti, cioè 20 contro a uno cienso e più 5 cose, fanno 20 ciensi e più 100 cose.

In the margin (corresponding to the “qui appresso” of the text) we find the scheme

100	100
per una cosa	per una cosa e più 5

In Canacci’s *Ragionamenti d’algebra* from c. 1490[9], we find a different schematization of the multiplication of polynomials [ed. Procissi 1954: 317]:

> Multiplicha 6 chose meno 3 p[er] n° vie 5 chose più 4 p numero dove achonceraj la multiplichatione chome una chasella eppoj inchominceraj dappie la multiplichatione dicendo meno 3 per numero vie 4 più p n° fa meno 12 e serba e ora faraj multiplichationi in note coe 6 chose vie 4 più per numero fa 24 chose e q.esta e la prima dipoj per la sechonda multiplicha 5 chose vie 3 meno 3 per numero fa meno 15 chose le q.ali tratte di 24 chose resta 9 chose e q.este ragiugni chon men 12 per numero serbato fai chose men 12 per numero e ora mulciplichereno 6 chose vie 5 chose che fa 30 censi e questo ragiugni chon 9 chose meno 12 per numero fa 30 censi e 9 chose chose

[8] This has a slight, but hardly more than slight, similarity to what Frits Staal [1995a: 78*f*] refers to as “artificial Sanskrit”, representing indeed a “kind of structural” deviation from what could be said in ordinary language.

[9] For this dating based on an autograph, see [Van Egmond 1983: 117].

meno 12 per numero per tale moltiplichatione

6 s m 3 p n°
5 s più 4 p. n°

---

30 censi p 24 s m 12 p n°
m 15 s

---

fa 30 censi e 9 s m 12, p n°

A similar use of marginal or otherwise separate repetitions of the rhetorical text in schemes or symbols is found in texts belonging to the Maghreb tradition and reflecting practices developed in the twelfth century [Abdeljaouad 2002: 9–14]; since Jacopo's algebra is derived from a hitherto unknown channel to the Arabic world and not from the Latin predecessors, Dardi's probably too, and since Canacci knows the meaning of Arabic terms never discussed in the Latin texts, the *abbaco* schematization and formal operations could well represent a borrowing from the Islamic world.[10]

More full-fledged and standardized schemes are found elsewhere – for instance in al-Samaw'al's writings[11], in the "matrix"-based of systems of linear equations in chapter 8 of the Chinese *Nine Chapters* [trans. Vogel 1968: 80*f*], and in India. Since Nesselmann sees the Indian methods as a first example of genuine symbolic algebra, I shall concentrate on an Indian example (hoping not to err too grossly because of my failing ability to read the original texts).

This example is the Bakhshālī manuscript – various types of equations or equation-analogues are discussed in [Hayashi 1995: 92–95], and more briefly in [Datta & Singh 1962: II, 30].[12] For instance, the equation system

$$\sqrt{x+A} = u \ , \ \sqrt{x-B} = v$$

appears as

0	*A yu*	*mū*	0	*sā* 0	*B*+	*mū*	0
1			1	1			1

However, as Hayashi [1995: 92] observes,

> [although,] Apparently, some of the expressions [...] have come very near the modern

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[10] Most likely from the Maghreb or al-Andalus – the schemes and abbreviations developed in the East by al-Samaw'al and others are of a rather different type. See, e.g., the various examples in [Rashed 1984] and [Rashed 1986: I].

[11] His sophisticated use of such schemes for the extraction of the fifth root of numbers is translated in [Rashed 1984: 102–108].

[12] Other Indian sources contain different kinds of schemes, but as far as I understand them they lend themselves to conclusions similar to those that can be drawn from this text.

> algebraic equations, [...] they are actually mere tabular representations of the numerical data, and are not equations to be the objects of operations.

Indeed, the scheme just given

> faithfully follows the verbal expression of the problem [...]
>
>> What number, when increased by five, has a square root? The same number, when decreased by seven, has a square root. What is that number? It has been questioned
>
> The answer to this problem is obtained not by means of the transformation of the equations, but according to the algorithm prescribed in Sūtra 50.

Once more, the schemes thus facilitates Neugebauer's "leichte Übersehbarkeit der Operationen". The other aspect – the linkage between scheme and algorithm – is no idiosyncrasy of the Bakhshālī manuscript: if the closed structure of the scheme is to serve more than visual transparency it is indeed as a support (in the likeness of a flow-chart) for an algorithm[13]. As pointed out by Hayashi et al [1990: 155], the scheme that allows Śaṅkara to express $\frac{4m^2-4}{4m^3-4m}$ as

4	0	4°	
4	0	4°	0

fails in such cases "where either more than one term of the same power, or an expression consisting of factors, occurs", for instance if the fraction to be expressed is $\frac{(4m^2-4)+(4m+1)}{m(2m-1)(2m+3)}$. We may still regard the schemes as an alternative, and thus an artificial language, and they certainly ask for operation at the level of symbols; but we are forced to admit that such schemes lack that recursiveness or possibility for embedding which characterizes any normal human language (creole languages tend to develop syntacticized relativization within a single generation or two from their emergence, see [Romaine 1988: 241–248]). This implies that schemes (like simple algorithms) are not fit for developing *new* levels of mathematical insight – the qualitative expansion of mathematical knowledge (for instance, how to generalize the methods used to solve second-degree equations to the third-degree case) has to build on natural-language reasoning (whether syncopated or not) coupled to wholly different (for instance, geometric) representations – neither an Indian scheme nor the traditional *al-jabr* algorithm

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[13] The example that may be most familiar to a modern mathematician may be the multiplication of matrices – or, if we think of the place-value notation for numbers as schematic representations of numbers (as did medieval Arabic writers), the algorithms for place-value arithmetic.

("halve the coefficient of the root, multiply it by itself ...") could be of the slightest use here.

In this respect, Viète has not gone significantly beyond the *Trattato dell'alcibra amuchabile* or the Maghreb writers – the only place where he *operates* on the level of symbols is where polynomial expressions enter into fractions as denominators. Bombelli had actually gone further when introducing algebraic parentheses (even embedded parentheses[14]) in his syncopated exposition (another illustration of the observation made above that the "beginning of formal, non-linguistic operations is no mere extension of the syncopation principle"). Descartes did not go much beyond *him* when introducing similar embeddings in a purely letter-symbolic notation, as illustrated by this formula from book II of his *Geometrie* – reproduced from the facsimile in [Smith & Latham 1954: 61, originally p. 325]

$$yy \;\infty\; \frac{\left.\begin{array}{l}-dekzz\\+cfglz\end{array}\right\}y \left.\begin{array}{l}-dezzx\\-cfgzx\\+bcgzx\end{array}\right\}y \left.\begin{array}{l}+bcfglx\\-bcfgxx\end{array}\right\}}{ezzz - cgzz.}$$

– in more familiar symbols $y^2 = \frac{(cfglz-dekz^2)\,y-(dez^2x+cfgzx-bcgzx)\,y+(bcfglx-bcfgx^2)}{ez^3-cgz^2}$. Manipulation of such symbolic expressions still did not replace ordinary-language reasoning, but within a century it it took over most of the former role of geometric representations. Within another century we reach Nesselmann, whose description of the actual situation was described above; within three we are already well beyond Russell & Whitehead and far beyond our topic.

Babylonia once more

Let us therefore return to the language of the Babylonian mathematical texts – more precisely, to the texts belonging to the time between 1800 and 1600 BCE (the second half of the Old Babylonian period, during which most of the extant mathematical texts were produced, including in particular Neugebauer's "series texts"). At first we may look at the first lines from two problems in almost purely

[14] In the printed work [Bombelli 1572] written ⌊...⌊...⌋⌋, in the manuscript (as rendered in illustrations in [Bombelli 1966] appearing as ⌊ ⌊ ⌋⌋ .

syllabic writing, belonging to the text BM 13901[15]:

Obv. II

Nº 12

27. a-šà *ši-ta mi-it-ḫa<-ra>-ti-ia ak-mur-ma* 21.40
The surfaces of my two confrontations I have accumulated: 21´40˝.

28. *mi-it-ḫa-ra-ti-ia uš-ta-ki-il$_5$-ma* 10
My confrontations I have made hold: 10´.

...

Rev. I

Nº 19

50. *mi-it-ḫa-ra-ti-ia uš-ta-ki-il$_5$-ma* a-šà^{lam} *a*[*k-mu*]*r*
My confrontations I have made hold: T[he] surface I have [accum]ulated.

First of all the terms have to be explained.

– a.šà, translated "surface", is a Sumerogram (a logogram of Sumerian origin) that corresponds to the Akkadian word *eqlum* (as can be seen in Rev. I 50, where the writing with a phonetic complement *-lam* stands for the accusative *eqlam*. It is the technical term for the area of geometric figures, in which function it is invariably written with the Sumerogram; problems dealing with real fields (for instance, those of the texts VAT 8389 and 8391, concerned with two fields that yield a rent) refer to these with a different and (to all we know) less adequate word, apparently in order to isolate the technical term from everyday use.
– *mitḫartum*, translated "confrontation", refers to the quadratic configuration parametrized by, and hence potentially identified with, the side[16]. The term is derived from the verb *maḫārum*, roughly "to confront an equal", and stands for "a situation characterized by the confrontation of equals".
– The enclitic particle *-ma* following after a verb (and then rendered as ":")

[15] Originally published in [Thureau-Dangin 1936], republished in [MKT III, 1–5]. I follow the text with parallel translation given in [Høyrup 2001: 165–174], using however Thureau-Dangin's system for transcribing sexagesimal place value numbers, in which ´, ˝, ... indicate decreasing and `, ``, ... increasing sexagesimal order of magnitude (10` thus stands for $10\cdot60 = 600$, 21´40˝ for $21\cdot60^{-1}+40\cdot60^{-2}$). Syllables in italics transcribe Akkadian phonetic (syllabic) writing, spaced writing is used for Sumerian word signs.

[16] *Our* square, like a Greek τετράγωνος, is supposed to *be* (e.g.) 9 m^2 and to *have* a side of 3 m. A Babylonian *mitḫartum*, like a Greek δύναμις, *is* its side and *has* an area. Whereas we think primarily of a square as what is contained by the quadratic frame, the Babylonian notion thus concentrates on this frame.

indicates consecution or consequence, and could be translated "and then" or "thus" (after a noun its function is that of providing emphasis, *X-ma* "that *X*").

– *kamārum*, translated "to accumulate", is a symmetric additive operation that may regard the measuring numbers of the entities that are added. It thus allows such concretely meaningless additions as those of surfaces and sides, of volumes and areas, or of men and working days.
– *šutakūlum*, translated "to make hold", is the causative-reciprocative stem of *kullum*, "to hold"; "to make [the measurable lines] *a* and *b* hold" means "to construct the rectangle contained (or held) by the sides *a* and *b*". Obviously the area of this rectangle will be $a \times b$; with utterly few exceptions, the computation of this area is considered inherent in the operation of "making hold" and is not made explicit. "Making *a* hold" is an ellipsis for "making *a* hold together with itself", and thus refers to the construction of a square with side *a*. Even in this case, the determination of the area is considered inherent in the operation.

When looking at the two problem statements we notice that both include the phrase *mi-it-ḫa-ra-ti-ia uš-ta-ki-il*$_5$*-ma*, "my confrontations I have made hold:". We should not be fooled by the identical formulations: they do not stand for identical operations. Both problems deal with two squares with sides $s_1 = 30'$ and $s_2 = 20'$, respectively (the implicit unit being the n i n d a n or "rod" of approximately 6 m). In problem 12, the phrase refers to the construction of the rectangle $\sqsubset\!\sqsupset(s_1,s_2)$, in problem 19 to the separate construction of the squares $\square(s_1)$ and $\square(s_2)$.

Such ambiguities do not agree with what we expect a technical terminology to achieve. However, the Babylonian reader of the text is not likely to have been puzzled. The clue to the interpretation is provided by the context of each phrase. In problem 12, the two areas have already been "accumulated" in line 27, and it would therefore make no sense to produce them in line 28. The appearance of a numerical result also indicates that a single rectangle is produced. Moreover, 30´ and 20´ were the standard sides of two-square (and one-rectangle) problems, and the value 10´ of the outcome confirms that we have to do with the standard rectangle $\sqsubset\!\sqsupset(30',20')$. In problem 19, on the other hand, no single result appears, and the "accumulation" later in the line shows that two areas must have been produced.

This concerned an almost completely phonetic text (only one word sign, a.šà, appears in the quoted passage). But the situation is certainly not better in the texts written exclusively with word signs, those which Neugebauer compares to modern symbolic algebra. This is illustrated by the series text YBC 4714, a catalogue of problem statements with appurtenant solutions but no procedure prescriptions.[17] We may pass over minor difficulties, for instance that context has to decide whether íb.si$_8$ 3.e means "the three confrontations" or "the third confrontation" (in this text type, íb.si$_8$ is used as a logogram for *mitḫartum*; in consequence, the distinction between the singular and the plural has disappeared from the written expression). Instead we shall look at problem 6:

Obv. II

7. a.šà íb.si$_8$ 3.e
The surface of 3 confrontations

8. *ù* íb.si$_8$ ⌞3].e
and the ⌞3] confrontations

9. g̃ar.g̃ar-*ma* 27.⌜50⌝
accumulated: 27`⌜50⌝.

10. íb.si$_8$ íb.si$_8$.ra
Confrontation to confrontation,

11. igi 17 gál ba.lal
The 17th part diminishes,

12. ½ nindan 1.e daḫ
½ nindan (being) appended to the 1st,

13. 3 nindan 2.e daḫ
3 nindan appended to the 2nd,

14. 2 nindan 3.e daḫ
2 nindan appended to the 3rd.

15. íb.si$_8$ 3.e en.nam
The 3 confrontations, what?

16. 25 nindan 1.e
25 nindan, the 1st.

17. 24 nindan 2.e
24 nindan, the 2nd.

18. 20 nindan 3.e
20 nindan, the 3rd.

Some supplementary explanations of the terminology are evidently needed:

– íb.si$_8$, as stated, is used in this text group as a logogram for *mitḫartum*, "the confrontation" (identified numerically, we remember, with the measure of

[17] First published in [MKT I, 487–492]. I follow the text with parallel translation given in [Høyrup 2001: 200–211].

the side of the square).

- g̃ar.g̃ar is a common logogram in mathematical texts for *kamārum*, "to accumulate".
- /.ra/ is the Sumerian dative suffix, serving here as a logogram for the preposition *ana*, "to".
- igi *n* gál is the traditional Sumerian term for the *n*th part of something.
- lal is a logogram for *maṭûm*, "to be(come) small(er)"; /ba./ is a Sumerian verbal prefix.
- daḫ is a logogram for *waṣābum*, translated "to append". This is another additive operation, asymmetric and applied only in situations where the addition is a concretely meaningful augmentation which conserves the identity of the entity that is augmented (in the sense in which the identity of *my* bank account is conserved when interest – in Babylonian *ṣibtum*, "the appended" – is added).

With this in mind we see that the problem deals with three squares, for which we know that

$$\square(s_1)+\square(s_2)+\square(s_3)+s_1+s_2+s_3 = 27`50 .$$

In line 11 we are furthermore told that the decrease from side to side is $^1/_{17}$, presumably $^1/_{17}$ of the first side; since nothing different is stated (as it is in other problems) this formulation *should* concern the decrease s_2–s_3 as well as the decrease s_1–s_2, possibly after a modification of the values of these; but that is obviously not the case according to the solution indicated in lines 16–18. If we look at line 12 we see, however, that s_1 is augmented (while conserving its identity) by $^1/_2$ nindan. Renaming this augmented side Σ_1 we can verify that Σ_1–s_2 = $^1/_{17}\Sigma_1$.

In lines 13–14 we see that even s_2 and s_3 are modified, $\Sigma_2 = s_2$+3 nindan, $\Sigma_3 = s_3$+2 nindan. However, Σ_1–Σ_2 is not the same as $^1/_{17}\Sigma_1$, and this modification can therefore only be meant to concern the second difference – but neither s_2, Σ_2, s_3 nor Σ_3 is divisible by 17, and no combination between any two of these gives the former difference of $^1/_{17}\Sigma_1 = 1^1/_2$ nindan. But as observed by Neugebauer in his edition, the *de*crease from s_2 to Σ_3 (from old to new value) coincides with the *in*crease from s_1 to Σ_2 (old to new again).

It is of course possible that a line has been dropped accidentally during copying; but no simple emendation of the text will eliminate the unexplained shifts between original and modified values. Whereas the syllabic text above was ambiguous in the same sense as most ordinary language, forcing the receiver of the message to rely in part on the context of terms and phrases for precise interpretation, the elliptic logographic text is even further removed from what we would expect from an exposition in genuine technical language, and even less adapted to operation directly at the level of signs.

The reason is that the Old Babylonian mathematical texts were not meant to stand alone. They were intended as a support for memory; the real medium for operations was neither the written language of the tablet nor the more complete and grammatical string of spoken words into which the logographic text could be expanded. As far as the "algebraic" texts are concerned, this medium was geometric – more precisely, the geometry of measured or measurable rectangular and square areas and their sides.[18] In order to see how it functioned we may look at the full text of BM 13901, problem 12:[19]

27. a-šà *ši-ta mi-it-ḫa<-ra>-ti-ia ak-mur-ma* 21.40
The surfaces of my two confrontations I have accumulated: 21´40˝.

28. *mi-it-ḫa-ra-ti-ia uš-ta-ki-il*$_5$*-ma* 10
My confrontations I have made hold: 10´.

29. *ba-ma-at* 21.40 *te-ḫe-pe-ma* 10.50 *ù* 10.50 *tu-uš-ta-kal*
The moiety of 21´40˝ you break: 10´50˝ and 10´50˝ you make hold,

[18] I shall not argue in detail for this reinterpretation of the terminology and the procedures in the present connection (a full presentation of the argument is found, e.g., in [Høyrup 2002a]). The need for second thoughts on the matter clearly follows from Neugebauer's explanation [1934: 70] of how, in the 1930s,

> wie man es [...] bei der Interpretation eines Textes zu machen pflegt. Man analysiert zunächst die Zahlen. "3 a.rá 2 6" ist offenbar eine Multiplikation, dann kommt "igi 6 gál 10", d.h. es wird das Reziproke gebildet: 6 = 10´. Schließlich kommt eine Subtraktion, die die Ziffern 10, 7 und 6,50 miteinander verknüpft in der Form 7–10´ = 6°50´.

(I have changed Neugebauer's notation into that used elsewhere in the article). This principle is useful for understanding that a.rá ("steps of") as well as *našûm* ("to raise") and *šutakūlum* ("to make hold") are multiplicative operations; but only a total reading of the corpus reveals that they are not synonymous but used for different purposes, and only a reading that takes the words between the numbers seriously and does not see them as mere black-box operators is able to disclose what they really stand for. This, of course, could not be done at the moment when Neugebauer, Thureau-Dangin and others cracked the terminology for the first time.

[19] Cf. note 15. I have silently corrected a calculational error due to mistake made on the counting board (see [Høyrup 2002b] and its consequences, since it is irrelevant for our present purpose, and also changed the translation slightly in agreement with considerations presented in [Høyrup 2002a: 25*f*].

30. 1.57.21.40.e 10 *ù* 10 *tu-uš-ta-kal* 1.40
1´57˝21‴40⁗ is it. 10´ and 10´ you make hold, 1´40˝

31. *lìb-bi* 1.57.21.40 *ta-na-sà-aḫ-ma* 17.21.40.e 4.10 íb.si$_8$
inside 1´57˝21‴40⁗ you tear out: 17˝21‴40⁗ makes 4´10˝ equilateral.

32. 4.10 *a-na* 10.50 *iš-te-en tu-ṣa-ab-ma* 15.e 30 íb.si$_8$
4´10˝ to one 10´50˝ you append: By 15´, 30´ is equal.

33. 30 *mi-it-ḫar-tum iš-ti-a-at*
30´ the first confrontation.

34. 4.10 *lìb-bi* 10.50 *ša-ni-im ta-na-sà-aḫ-ma* 6.40.e 20 íb.si$_8$
4´10˝ inside the second 10´50˝ you tear out: by 6´40˝, 20´ is equal.

35. 20 *mi-it-ḫar-tum ša-ni-tum*
20´ the second confrontation.

Another five new technical terms and phrases turn up here:

– *bāmtum* is the “natural” or “necessary” half of something, a half that in the nature of things could not have been anything else (for instance, that half of the base of a triangle that has to be multiplied by the height in order to give the area). The translation “moiety” is meant to distinguish it from the ordinary half *mišlum*.

– *ḫepûm*, translated “to break”, is used in mathematical texts exclusively when a “moiety” is produced.

– *nasāḫum*, translated “to tear out”, is a concrete subtraction, the removal of part of an entity. It is thus the reverse operation of appending and identity-conserving in the same sense.

– the genitive of *libbum*, originally “heart” or “bowels”, is used in a weaker sense approaching “inside”, indicating that an operation concerns the “body” of an entity. We may thus “append to” or “tear out inside” an area, but we raise a prismatic base “to”, not “to inside” the height when calculating the area.

– *Q*.e *s* íb.si$_8$, translated “by *Q*, *s* is equal”, means that *s* is the side of the square area *Q* – /.e/ being the locative-terminative Sumerian suffix, /íb./ a grammatical mark of finiteness merged with a pronominal element, and the verb si$_8$ meaning “to be equal”.

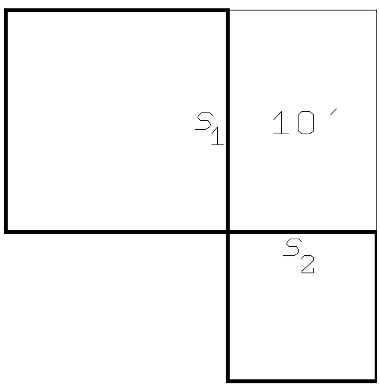

Figure 1. The initial situation of BM 13901 #12.

The situation is shown in Figure 1: The sides s_1 and s_2 of two square contain a rectangle, whose area is 10´ (line 28). We also know (line 27) that the sum of the two square areas is 21´40˝. The first step undertaken (lines 29–30) is that this sum is bisected (with result 10´50˝), and the square on it constructed (result 1´57˝21‴40⁗). The reason for this step is revealed in line 30, when the square on the rectangular area is constructed (result 1´40˝). In symbols, this square is $\square(\sqsubset\!\sqsupset(s_1,s_2))$, but the calculator knows that the square on the rectangle is numerically the same as the rectangle contained by the squares on the sides,

$$\square(\sqsubset\!\sqsupset(s_1,s_2)) = \sqsubset\!\sqsupset(\square(s_1),\square(s_2)) \ .$$

This rectangle (whose sides are $L = \square(s_1)$ and $W = \square(s_2)$) is shown in Figure 2. Its area is known to be $10'^2$ = 1´40˝, the sum of the sides is $L+W = \square(s_1)+\square(s_2)$ = 21´40˝. Thereby we have been brought to one of the standard problems of Old Babylonian second-degree “algebra”, and the procedure employed to solve it is the standard one. At first, the sum of the sides is bisected, and the square on the resulting moiety is constructed. This is the step that is performed in line 29. As can be seen, this square exceeds the area of the rectangle by the square $\square(\delta)$, δ being the deviation of L and W from their average μ – for those who are not accustomed to seeing such relations from diagrams they follow from these calculations:

$$L-\mu \ = \ L-\tfrac{L+W}{2} \ = \ \tfrac{L-W}{2} = \delta \ , \ \ \mu-W = \tfrac{L+W}{2}-W \ = \ \tfrac{L-W}{2} = \delta \ .$$

“Tearing-out” the rectangle 1´40˝ inside $\square(\mu)$ thus leaves $\square(\delta)$, which is found to be 17˝21‴40⁗, alongside which 4´10˝ = δ “is equal”. When δ is appended to one of the two copies of μ (in the diagram the horizontal side of the large square) we get $L = \square(s_1) = \mu+\delta$ = 10´50˝+4´10˝ = 15´, alongside which s_1 = 30´ “is equal”. When δ is torn out from the other copy of μ (the vertical side in the diagram), we get $W = \square(s_2) = \mu-\delta$ = 10´50˝–4´10˝ = 6´40˝, alongside which s_2 = 20´ “is equal”.

This geometric level is the level of “real” operations, of which the verbal description in the texts – whether phonetic or logographic – is nothing more than a description.[20] The geometric level is the level of mathematical reasoning

[20] Also “real” is of course the level of numerical computations. Even these, however, are made outside the text and then described in it, either mentally, on pads for rough calculation by means of tables, or on a counting board (mostly in combination, we may suppose). However, the numerical level may be claimed to be slightly less real than the geometric representation, in the sense that all “algebra” texts beyond the first degree were constructed backwards from known results (which explains that the present text ends up with the correct result in spite of the wrong intermediate result: when finding the square root of 17˝46‴40⁗ (which should have been 17˝21‴40⁗) the calculator writes

and the level where the correctness of the procedure becomes evident. In some cases, texts choose between technically synonymous words for the same operation, perhaps from general semantic connotations (for instance “tearing out” from areas but “cutting off” / *ḫarāṣum* from lines); the terminology, though fairly standardized, is always *less so* than the geometric procedure. The relation between geometric representation and textual description is thus rather similar to that between a modern proof by means of symbols and the verbal exposition of this proof in the lecture room. The modern mathematician will evidently point to the formulae on the blackboard or screen; but arguments can be given that the Old Babylonian schoolmasters also had some kind of diagram at their disposal when they taught these matters.[21]

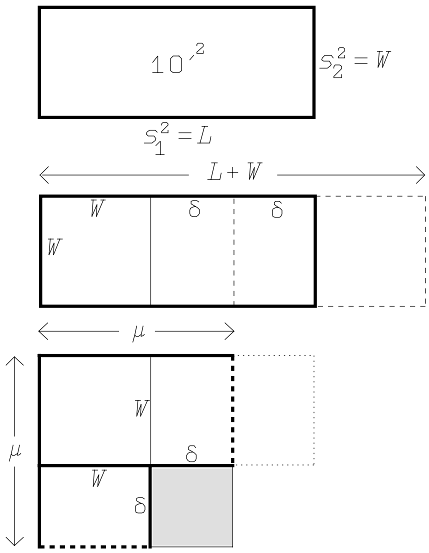

Figure 2. The new rectangle of BM 13901, and the procedure.

All in all we may thus conclude that the Old Babylonian “algebraic texts” (those texts where something different could perhaps have been expected) do *not* make use of an artificial language in any proper sense, in spite of their highly stereotyped character and their logograms. Neugebauer was not wrong when read closely, but reading his remarks about logograms as symbols *as if* his concept of symbols coincided with Nesselmann’s is certainly fallacious.

If we follow those structuralists that spoke of every ordered domain of operations as a “language” (for instance, the Lacan school) we might still regard the set of operations as constituting an artificial language, further removed indeed from natural language than most of what we normally consider under this heading. In the present problem we even notice the phenomenon of embedding,

what he knows should result, namely 4´10´´, the square root of 17´´21´´´40´´´´.

[21] It may for instance be noticed that BM 85200+VAT 6599, a long “algebraic” text about “excavations”, that is, prismatic volumes, groups problems together which deal with the same configuration, irrespective of the being of the first, second, or third degree, and separates problems widely that are closely related mathematically (for instance, the third-degree problems)

namely of the standard problem $\sqsubset\!\sqsupset(s,t) = \alpha$, $s+t = \beta$.[22] This, however, is an extension of the argument that I shall pursue no further, apart from pointing to the related analysis of ritual order in [Staal 1995b].

Leaping backwards

Instead we may leap backwards in time to the very beginning of Mesopotamian literacy and written numeracy – that is, to the second half of the fourth millennium BCE.

Long before that, accounting had been made in the Syro-Iraqo-Iranian region (probably for redistributive purposes) by means of small counters of burnt clay shaped as spheres, cones, cylinders, etc., probably tokens representing standard containers of grain and oil, heads of cattle, and other staple goods. In the fourth millennium, as redistribution was transformed into regular payment of tribute or taxes to a central temple system, this token-based system was integrated in more complex procedures, the tokens being for instance enclosed in sealed clay containers ("bullae") apparently functioning as bills of lading.[23] Often, it was indicated on the surface of the bulla which tokens were enclosed; in this way the document could be "read" without being destroyed.

Precise dating of the various steps is difficult, but it seems to be a secondary discovery that this marking of the surface of the bulla made it possible to dispense with the contents and to replace the bulla by a piece of flattened clay. Soon, in any case, flattened rectangular clay tablets appear in Uruk in southern Iraq carrying not only impressed metrological signs corresponding functionally to the tokens (in some cases indubitably looking like depictions of familiar tokens) but also signs for words; from the very beginning a large repertoire of signs is in use, and we are thus confronted not with any gradual development but with an invention, a planned creation. The logograms of the later cuneiform writing developed from these original word signs (and so did the syllabic signs, when the word signs started being used for writing of grammatical elements according to the rebus principle). The Egyptian and proto-Elamite writing systems, created very soon afterwards in regions that were in contact with the Uruk region, may

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[22] For further exploration of the nesting of procedures or algorithms in Old Babylonian mathematics, see [Ritter, forthcoming].

It seems implausible that any kind of mathematics developing on the basis of a metrology or number system organized in smaller and larger units can avoid being structured "linguistically" in this metaphorical sense.

[23] For this, see for instance [Schmandt-Besserat 1992].

have been created as emulations of proto-cuneiform.

This beginning of genuine writing took place during the so-called Uruk IV period (presumably c. 3300–3100 BCE). It is interesting in two respects for the question of artificial languages.

Proto-cuneiform, indeed, was not created in an attempt to render spoken language, at least not syntactically, probably not even at the level of vocabulary.

The former claim is easy to defend, and I shall leave it aside for a short while. The latter is less manageable, considering that we have no access to the vocabulary of the Uruk inhabitants beyond their written documents. I shall start by that.[24]

Around the mid-fourth millennium, as falling water levels made irrigation agriculture possible, southern Mesopotamia (the area of Uruk, the future Sumerian region) experienced a violent increase in population. Most likely, there was a strong immigration from the mountain regions to the east (which the same climatic change will have made inhospitable), but continuity in temple architecture since the earliest fifth millennium demonstrates that the culturally hegemonic stratum in the Uruk region was indigenous – which means that its language will have left any pidgin-like structure far behind long ago.

Composite proto-cuneiform signs, on the other hand, were constructed according to principles that are quite similar to what is familiar from pidgin and creole languages. As an illustration, Figure 3 confronts a list of composite words from Tok Pisin, a mature Pidgin used in Papua New Guinea, and a sequence of composite proto-cuneiform signs.

Interesting in this connection is that the immigrants were likely to be enslaved. A favourite theme on the seal of officials, existing in many variants, shows a high priest looking at overseers beating up pinioned prisoners; the signs for male and female slaves, moreover, consist of the signs for a male or female combined with a stylized depiction of the mountain ridge to the east. The region was probably as rich then as later in languages, and the immigrants can therefore be supposed to have spoken many different tongues. All in all, the situation is likely to have been similar to that of the plantation economies of the early Modern epoch, and hence to have been an ideal cradle for the creation of a pidgin and the ensuing emergence of a creole – and many features of Sumerian, when this language surfaces in the written documents from around 2600 BCE onward,

[24] For this I draw heavily on [Høyrup 1992], omitting most of the detailed arguments given there.

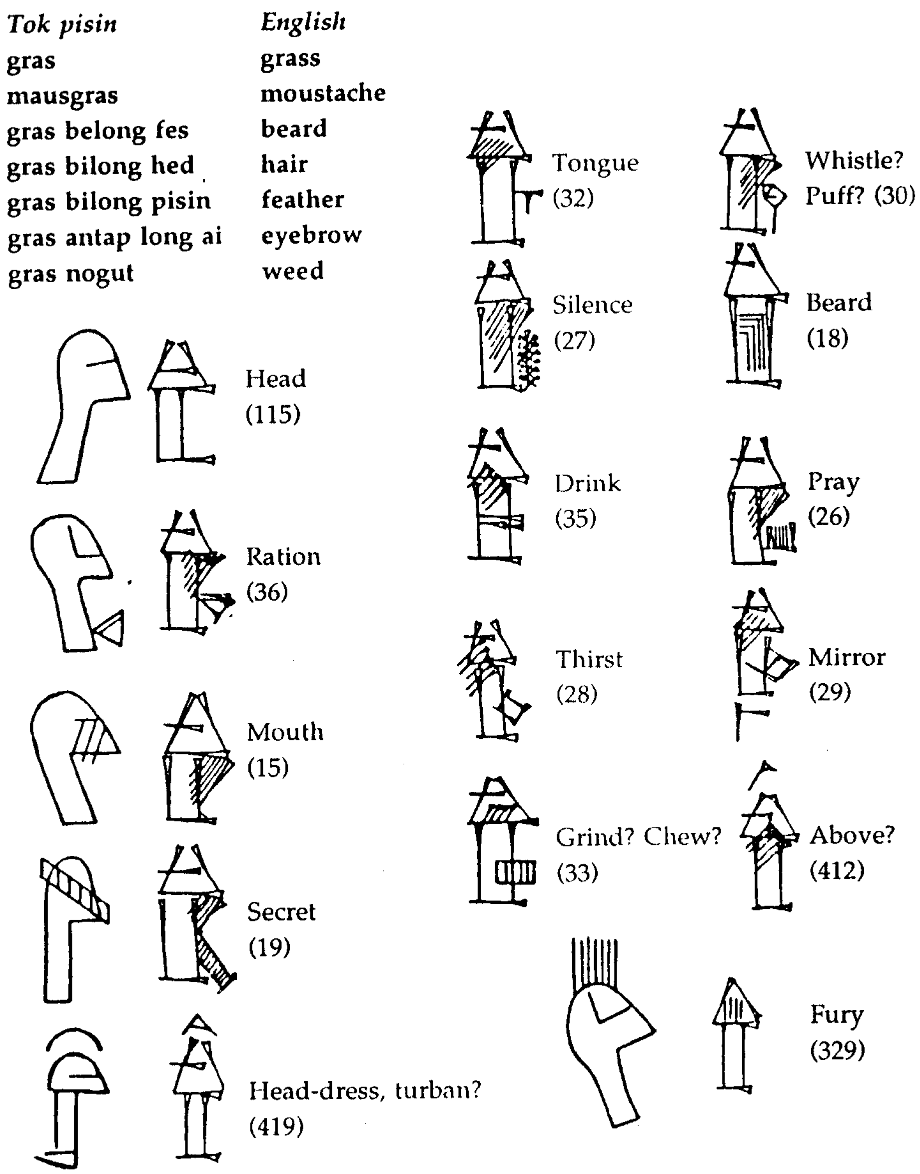

Figure 3. A sequence of compounds containing the constituent "gras" from Tok Pisin with English translations, borrowed from [Romaine 1988: 35], and a sequence of proto-cuneiform signs and their third-millennium cuneiform descendants derived from the sign "head", copied from [Labat 1948, *passim*] (some of the signs have not been located in the Uruk material and are therefore only shown in their third millennium shape). The meanings are derived in part from later logographic applications of the signs, in part from the signs themselves. That the result may be only approximate in certain cases is exemplified by the sign for ration apportioning, which according to its later use might seem to mean simply »eat«.

are indeed similar to those that develop with time within creoles.[25]

[25] One may wonder that the language of the slaves should take over if originally not that of the masters, but this is a common development. Up to 70% or more of the basic

Masters listening to the creole of their slaves (or those who know English as a primary or a secondary language and see written Tok Pisin) are likely to misunderstand much of the real structure of the language (the creolist literature abounds in examples) or to recognize only primitive congregation of words. But this is sufficient; knowledge of the conjectural creole will have been enough to suggest the principles that were used in the creation of the composite signs.

Alternatively, it is a possibility that the same universal cognitive resources as are used in the development of a pidgin were applied by the inventors of the Uruk script.[26] What can be safely excluded is that the script was a direct expression of an existing pidgin or creole: the conceptual domain of which administrative documents make use is very different from that which structures the daily life of slaves and which is expressed in the language they create. At the level of vocabulary, the written language of Uruk IV was certainly created with purpose as an artificial language.

At the discursive and syntactical level, the artificial character of the Uruk IV script is much more easily argued. Since the argument has been well made by other scholars (not least [Green 1981], [Nissen et al 1993]), I shall restrict myself to reporting their conclusions.

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vocabulary of a creole is taken over from the superstrate (see the Tok Pisin sequence, in which it should perhaps be pointed out that "antap" derives from "on top", "pisin" corresponds to "pigeon", and "nogut" to "no good"); moreover, the kids of the masters (that is, the future masters) are taken care of by slave women rather than by their mothers. Within a few centuries, Afrikaans has acquired many of the characteristic features of a creole (for instance, losing gender), in spite of a metropolis where Dutch was spoken [Romaine 1988: 54*f*]. In the Uruk region, no reservoir for decreolization will have been at hand.

[26] However, as observed in [Høyrup 1992: 33 n. 30],

> the conscious construction of an extensive and elaborate system is very different from the accumulation of individual communicative emergency solutions which ends up as a pidgin; it is thus not very likely that even the same fundamental cognitive processes would produce structurally similar results in the two situations. Emulation of the structure of the final outcome of pidginization as this is conceived by outside observers, on the other hand, cannot avoid to produce at least superficially similar patterns, even though the cognitive process is now different.
>
> The point where similarity between cognitive processes certainly plays a role is in reception: The reason that the proto-cuneiform script can function as a communicative system (within a well-defined context, that of bureaucratic procedures) will not be different from the reason that an early pidgin can function (even this within a restricted context).

Upwards of 10% of the extant texts and text fragments from Uruk IV and the ensuing Uruk III phase are lexical list that served for teaching the script [Nissen 1981: 101]. Some of the others are small tags carrying a few non-metrological signs not known from other contexts, probably personal names and "probably attached with a string to containers or other items, stating the proprietor or receiver of such goods" [Nissen et al 1993: 20]. The latter of these types is not interesting for our present purpose; we shall return to the lexical lists in a moment.

The remaining texts – the immense majority – are simple or composite accounts. Again, the simple accounts are not informative – see the example in Figure 4. It may mean as much as "Two sheep delivered to/from the temple of Inanna", and all we can say is that the signs are not ordered in a way that seems to reflect any attempt to render a spoken sentence.

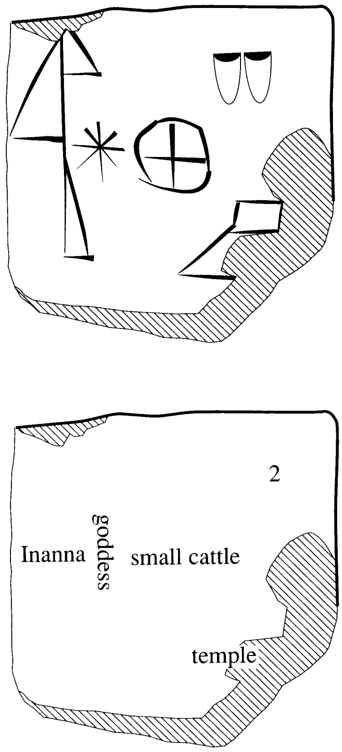

Figure 4. A simple accounting text from Uruk III, after [Nissen et al 1993: 21]. I have turned the copy back to its original position – the publication shows it turned 90° counterclockwise, in agreement with the later orientation of cuneiform tablets.

The composite accounts tell us more because they possess an order of their own. A crude example is shown in Figure 5, whose obverse lists four separate entries, the edge another one, and the reverse some total (apparently for these four and several other transactions, or for a quintuple repetition of the transactions on the obverse). Other composite accounts are much more sophisticated; they may combine ordinary barley with malted barley for each of several recipients, and they may indicate the occasion for the transactions and/or the responsible official; but the principle remains the same, and the order of the documents still corresponds to that of accounting ledgers or statistical tables, not to that of any possible spoken language.

The Uruk script was thus really an artificial language, meant to express bureaucratic order, and certainly also facilitating further development of this order (composite accounts are much more numerous and more developed in Uruk III than in Uruk IV, and later epochs present us with developments which were only (if at all) equalled in quite recent times.[27] In particular we may take note of the incipient use of the "cartesian product" in the ordered listings of (e.g.) barley and malt for each of several recipients. In the so-called "profession list", a lexical list enumerating the professions of the city state, the same principle is more clearly visible: for each of those professions where it is relevant (thus not, e.g., for the head of the assembly) it lists the senior responsible, overseers and common workers.

Some of the Old Babylonian catalogue texts make use of the same principle in up to four dimensions (the excerpt from YBC 4709 discussed above by Neugebauer makes use of two dimensions, combining the variations +/− and $x^2/2x^2$). Through the tables of the Late Babylonian astronomers this kind of tabular order became known to classical antiquity; since astronomical tables were still the most orderly tables known to the literate in the late Middle Ages and the Renaissance, present-day bureaucratic order may own much more than we usually think to the artificial language created in Uruk IV.

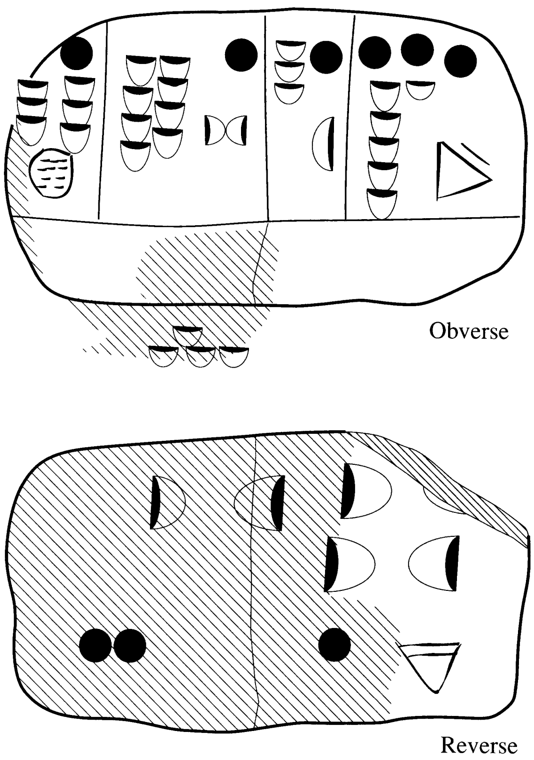

Figure 5. A crude composite account for grain products. The reverse is likely to contain a summation, and indicates that grain rations ( ) are dealt with. After [Nissen et al 1993: 31].

This observation is not the only one we can make on the lexical lists. We may also link them to Luria's distinction [1976: 48*ff*] between "situational thinking" and "categorical classification". The latter type is abstract and analytical in the sense that it looks at things in isolation from the wider situation in which they occur or are used. The former, instead, understands the world synthetically, through total situations; it is "economical" for a person whose life is made up

[27] Writing within the confines of the German war economy, Nikolaus Schneider [1940: 5] would still say about the global societal accounting of the 21st-century Ur III economy "daß wir sie sogar heute als überspitzt bezeichnen müssen".

by fixed and recurrent situations.[28] Categorical classification, on its part, is the adequate way to orient oneself in a life world which is not organized in this stable way. Apart from professions, lexical lists enumerate, for instance, various types of vessels; objects made from wood; metal objects; geographical names; etc. They do not put the ox, the plough, the ploughman and the vessel containing the grain together. In this way the training of the script through the lexical lists will also have been a training in conceiving the world in what we would consider a "modern" way – certainly an adequate way to see it for temple managers who had to think of a plough both as an object to be constructed in the workshop, as an agricultural tool, and perhaps as an object of taxation.

Around or shortly before 2600 BCE, the first royal inscriptions turn up, and before 2500, when a profession of scribes distinct from the stratum of managers manifests itself in the sources, the earliest literary texts (and the earliest mathematical problems that are not mere model accounting documents) turn up. By then, the script was used to render spoken language (doing so at first rather imperfectly, but that is not important), and its character of an artificial language becomes less conspicuous. Remembering the beginnings, however, we may try to stay aware to the fact that writing was always an artificial language as long as language proper is understood as spoken language, and that writing was originally invented on purpose as a much more artificial language than it developed into being. The main contribution of the ancient Mesopotamian world to the development of artificial languages was certainly its invention of writing.

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[28] Luria [1976: 55] mentions a peasant who wants to categorize a boy together with a collection of tools because the boy will be useful to fetch the tools for those adults who are to work with them.

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